Quantum metrology with Bose-Einstein condensates

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Abstract. We show how a generalized quantum metrology protocol can be implemented in a two-mode Bose-Einstein condensate of n atoms, achieving a sensitivity that scales better than $1/n$ and approaches $1/n^{3/2}$ for appropriate design of the condensate.

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In a separate paper elsewhere in this volume [1], we showed that two-body couplings between the n qubits that make up the quantum probe in a parameter-estimation scheme can lead to measurement sensitivities that scale as $1/n^{3/2}$ even when the initial state of the probe is unentangled. A Bose-Einstein condensate (BEC) is a physical system in which effective two-body couplings exist between all the atoms in the condensate. We examine in this contribution how a BEC of n atoms can be turned into a quantum probe to measure a parameter with a sensitivity scaling that approaches $1/n^{3/2}$.

The many-body Hamiltonian, in second-quantized notation, for a dilute Bose gas in which the inter-particle spacing is much larger than the scattering length, a, is [2, 3, 4, 5]

$$H = \int d\boldsymbol{r}\left(-\frac{\hbar^2}{2m}\nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi} + V(\boldsymbol{r})\hat{\psi}^\dagger\hat{\psi} + \frac{g}{2}\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}\right), \tag{1}$$

where $\hat{\psi}^\dagger$ and ψ are creation and annihilation field operators that obey bosonic commutation relations. The gas is in a trapping potential $V(\boldsymbol{r})$ and the coupling constant $g = 4\pi\hbar^2 a/m$. In a BEC at zero temperature almost all the atoms are in the ground state; we can write $\hat{\psi}^\dagger(\boldsymbol{r}) \simeq \psi_n(\boldsymbol{r})\hat{a}^\dagger$, where $\psi_n(\boldsymbol{r})$ is the single-particle ground-state wave function and $\hat{a}^\dagger$ is operator that creates atoms with this wave function.

So far we have assumed that all the atoms in the BEC are in the same atomic state, but to use the atoms as components of a quantum probe, we want them to be two-level systems or qubits. We thus consider two-mode BECs in which the atoms can occupy one of two internal states, labeled $|1\rangle$ and $|2\rangle$. These two states are typically hyperfine levels of the atoms. In practice, the atoms are cooled to form the BEC while they are all in the same internal state, and then an external field is used to drive transitions between the two levels to achieve the desired coherent superposition of the two levels. We assume that

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the collisions between the atoms are elastic. We also assume that the the second internal state is chosen so that it sees the same trapping potential $V(\boldsymbol{r})$; therefore, for short times, the wave functions for atoms in both states $|1\rangle$ and $|2\rangle$ can be approximated by the same wave function $\psi_n(\boldsymbol{r})$. Using these assumptions can write down the Hamiltonian for the two-mode BEC as

$$\begin{aligned} H &= \int d\boldsymbol{r}\Big[-\frac{\hbar^2}{2m}|\nabla\psi_n(\boldsymbol{r})|^2\left(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2\right)+V(\boldsymbol{r})|\psi_n(\boldsymbol{r})|^2\left(\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2\right) \\ &\quad +\frac{g_{11}}{2}|\psi_n(\boldsymbol{r})|^4\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1+\frac{g_{22}}{2}|\psi_n(\boldsymbol{r})|^4\hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_2\hat{a}_2+g_{12}|\psi_n(\boldsymbol{r})|^4\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2\Big] . \end{aligned} \tag{2}$$

To make the connection to the generalized quantum metrology protocols described in [1, 6, 7], we define the following two operators, $\hat{n}\equiv\hat{a}_1^\dagger\hat{a}_1+\hat{a}_2^\dagger\hat{a}_2$ and $\hat{J}_z\equiv(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)/2$. The operator $\hat{n}$ is simply the number operator; it gives the total number of atoms in the BEC, which is the relevant metrological resource. Treating $\hat{n}$ as a c-number, the Hamiltonian becomes

$$H=H_0+\gamma_1 n\hat{J}_z\eta+\gamma_2\hat{J}_z^2\eta \ , \tag{3}$$

where $(g_{11}-g_{22})/2\equiv\gamma_1$, $(g_{11}+g_{22})/2-g_{12}\equiv\gamma_2$ and $\eta=\int d\boldsymbol{r}|\psi_n(\boldsymbol{r})|^4$.

The first term, H_0, of the Hamiltonian (3) can be ignored, because it acts identically on atoms in states $|1\rangle$ and $|2\rangle$. Since the initial states of the probe that we consider are of the form $(\alpha|1\rangle+\sqrt{1-\alpha^2}|2\rangle)^{\otimes n}$, the effect of H_0 is only to produce an overall phase in the time evolved state of the probe. In the next two terms, we have both nJ_z and J_z^2 couplings that suggest that we might be able to measure the constants γ_1 and γ_2 with an accuracy that scales as $1/n^{3/2}$ [1, 6, 7]. This assumes that the factor η has no dependence on n. In fact, η is inversely proportional to the effective volume occupied by the ground-state wave function. Adding more atoms to a harmonically trapped BEC spreads out the wave function because of the repulsive scattering, thereby reducing η as n increases. To pin down the exact scaling of the measurement accuracy with n, we need to know how η behaves as a function of n.

Scaling of the measurement uncertainty

Since we first create a BEC of n atoms all in hyperfine state $|1\rangle$, before putting them in a superposition of states $|1\rangle$ and $|2\rangle$, we can focus on the dependence of η on n for a single-mode BEC of atoms in state $|1\rangle$. An obvious strategy to suppress the n dependence of η is to constrain the BEC within a hard-walled trap so that it cannot expand as more atoms are added. BECs effectively confined to two or one dimensions and held in power-law trapping potentials along the effective dimensions are the sort that are found in real experiments. Thus we look at the dependence of η on n for a BEC in d dimensions, referred to as *longitudinal* (L) dimensions, and held in a trap along those dimensions of the form $V_L(\boldsymbol{r})=\frac{1}{2}kr^q$, where q is a positive integer. There are $D=3-d$ tightly confined, *transverse* (T) dimensions along which we take the trapping potential to be harmonic, i.e., $V_T(\boldsymbol{\rho})=\frac{1}{2}m\omega_T^2\rho^2$. The bare, one-atom ground-state half-widths along the longitudinal and transverse dimensions are denoted by $r_0 \equiv (\hbar^2/mk)^{1/(q+2)}$ and $\rho_0 \equiv (\hbar/2m\omega_T)^{1/2}$, where r_0 is an approximate expression that scales properly with the atomic and trap parameters.

We define two critical atom numbers, n_L and n_T, as the number of atoms in the BEC that makes the scattering energy comparable to the longitudinal and transverse kinetic energies:

$$n_L \equiv \frac{r_0}{a}\left(\frac{\rho_0}{r_0}\right)^D, \qquad n_T = \frac{\rho_0}{a}\left(\frac{r_0}{\rho_0}\right)^{d(q+2)/q}. \tag{4}$$

Here a is the scattering length of the atoms in state $|1\rangle$.

When $n \ll n_L$, the scattering energy can be ignored, and the ground-state wave function is simply the ground-state solution for the trapping potential $V_L(\boldsymbol{r}) + V_T(\boldsymbol{\rho})$, i.e., a product wave function independent of n, $\psi_n(\boldsymbol{r},\boldsymbol{\rho}) = \phi(\boldsymbol{r})\chi(\boldsymbol{\rho})$. In this case η is a constant and the uncertainty in an estimate of γ_1 or γ_2 scales as $1/n^{3/2}$.

We are mainly interested in the regime of atom numbers satisfying $n_L \ll n \ll n_T$. For a 1D trap, typical trap parameters can make $n_L \sim$ 1–10, so it only takes a few atoms to make the scattering energy important along the longitudinal direction. As long as $n \ll n_T$, however, we can approximate the ground-state wave function to be a product, $\psi_n(\boldsymbol{r},\boldsymbol{\rho}) = \phi_n(\boldsymbol{r})\chi(\boldsymbol{\rho})$, where $\chi(\boldsymbol{\rho})$ is the ground-state, Gaussian wave function for the harmonic transverse trapping potential, and $\phi_n(\boldsymbol{r})$ is the ground-state solution for a d-dimensional, time-independent Gross-Pitaevskii equation that includes the longitudinal trapping potential $V_L(\boldsymbol{r})$ and the scattering term. For a product wave function, we have $\eta = \eta_L\eta_T$, where $\eta_T = \int d\boldsymbol{\rho}\,|\chi(\boldsymbol{\rho})|^4 = 1/(2\sqrt{\pi}\rho_0)^D$. To find η_L we assume that $n \gg n_L$ and solve the Gross-Pitaevskii equation in the Thomas-Fermi approximation, which ignores the longitudinal kinetic energy. Using the Thomas-Fermi wave function along the longitudinal dimensions and its normalization condition, we get

$$\eta = \eta_T\eta_L = (\alpha_{q,d}/\rho_0^D r_0^d)(n_L/n)^{d/(d+q)}, \tag{5}$$

where $\alpha_{q,d}$ is a geometrical factor of order unity that depends on q and d but not on n. Numerical computation of η using the full, 3D Gross-Pitaevskii equation indicates that this expression for η is quite accurate in the regime $n_L \ll n \ll n_T$, in spite of the several approximations that went into obtaining it [8].

For n between n_L and n_T, the effective n-dependence of the last two terms in Eq. (3) is $n^{2-d/(d+q)} = n^{(d+2q)/(d+q)}$, and the measurement uncertainties in γ_1 and γ_2 scale as

$$\delta\gamma_{1,2} \sim \frac{1}{n^\xi}, \quad \text{where} \quad \xi = \frac{d+3q}{2(d+q)}. \tag{6}$$

We find that a 2D BEC ($d = 2$) in a harmonic trap ($q = 2$) matches the $1/n$ scaling and a 1D BEC in a harmonic trap betters this scaling. Two-D and 1D BECs in harder traps ($q > 2$) perform even better with respect to the scaling of the measurement uncertainty.

If $n > n_T$ we cannot meaningfully consider the BEC to be lower dimensional. For a BEC in a 3D spherically symmetric trap, $n_L = n_T = r_0/a$ and $\xi = (3+3q)/(6+2q)$, which becomes $\xi = 9/10$ for a 3D harmonic trap. This is worse than the $1/n$ scaling, but better than the shot-noise-limited scaling of $1/\sqrt{n}$. If the trap is anisotropic and the

BEC is cigar-shaped, then assuming that the trap is harmonic along the two transverse dimensions, we find $\xi = (4q+1)/(4q+2)$, which reduces to $\xi = 9/10$ if the trap is harmonic along the longitudinal dimension as well. Thus we see that for doing better than the $1/n$ scaling when $n > n_T$, we have to use harder traps in all three dimensions.

A good candidate for implementing the generalized metrology protocol is a Bose-Einstein condensate made of rubidium (^{87}Rb) atoms [9]. Typically, the $|F=1; M_F = -1\rangle \equiv |1\rangle$ is trapped and cooled to the condensation point. Once the atoms in $|1\rangle$ have formed in the condensate ground state, a two-photon drive can be applied that couples the $|1\rangle$ state to the $|F=1; M_F=-1\rangle \equiv |2\rangle$ state. The s-wave scattering lengths for the three processes, $|1\rangle|1\rangle \to |1\rangle|1\rangle$, $|2\rangle|2\rangle \to |2\rangle|2\rangle$, $|1\rangle|2\rangle \to |1\rangle|2\rangle$ are nearly degenerate for ^{87}Rb, with the ratios $\{a_{22} : a_{12} : a_{11}\} = \{0.97 : 1 : 1.03\}$. This means that γ_2 is very close to zero for ^{87}Rb, and so we realize the generalized quantum metrology protocol with just the $\gamma_1 n \eta J_z$ coupling. We look at the practical considerations and limitations of a realistic metrology experiment using a BEC in [8].

The quantity that is measured in our metrology protocol is essentially a constant scattering length. Estimating a constant using sophisticated quantum measurement schemes is sometimes interesting, but in this only as a proof of principle. For our proposal using a BEC, one possibility is to work around a broad Feshbach resonance that makes the scattering lengths sensitive to external magnetic fields. We might then be able to use our scheme for high-precision magnetometry.

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